

Exotic Hill problem : Hall motions and symmetries

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Our previous study of a system of bodies assumed to move along almost circular orbits around a central mass, approximately described by Hill's equations, is extended to “exotic” [alias non-commutative] particles. For a certain critical value of the angular velocity, the only allowed motions follow the Hall law. Translations and generalized boosts span two independent Heisenberg algebras with different central parameters. In the critical case, the symmetry reduces to a single Heisenberg algebra.

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I. INTRODUCTION

The Hill problem arises as an approximation for nearly circular trajectories to Newton's gravitational equations written in rotating coordinates for bodies moving around a central mass. The original example is provided by the Moon-Earth-Sun system [1]. Later, Hill's equations have also been applied to stellar dynamics [2, 3], with a “star cluster” replacing Moon and Earth, and the “Galactic Center” playing the role of the Sun.

For the center-of-mass $\mathbf{r} = \sum_a m_a \mathbf{r}_a / m$ where $m = \sum_a m_a$, the inter-particle interactions cancel, leaving us, in the planar case, with the equations [4],

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - 3\omega^2 x &= 0, \\ \ddot{y} + 2\omega\dot{x} &= 0, \end{aligned} \quad (1)$$

where ω is the angular velocity $\omega^2 = GM/R^3$ for a circular Keplerian trajectory with radius R . Note here the “only- x ” anisotropic oscillator term, which is the remnant of the centrifugal and Newtonian forces under linear approximation.

The general solution of eqn. (1) is a combination of simply “Hall” motions of a guiding center, combined with elliptic motions around the guiding center [4].

The decomposability into center-of-mass and relative coordinates relies on having an Abelian symmetry made of (generalized) translations and boosts, characteristic for Galilean symmetry [5, 6]. These symmetries are also given by the solutions of eqn. (1); the conserved quantities associated with them span two copies of Heisenberg algebras with central parameters $\mp 2/m\omega$ [4].

The velocity-dependent terms in eqn. (1) come from the Coriolis force, and are analogous to a uniform mag-

netic field. Whence the similarity to the Landau problem and to the Hall effect.

On the other hand, some insight to the Hall effect could be gained by assuming that the charged particles admit an additional “exotic” structure, which makes the coordinates x and y non-commuting [7].

In this Note we generalize Hill's equations to exotic particles, providing us with a combination of the Hill results in [4] with the Hall-type behavior found before in the non-commutative Landau problem [8].

Our main result is that, for the critical value $\omega = \omega_c$ of the rotation i.e. of the radius, the only allowed motions are those which satisfy the Hall law.

Our basic tool is the generalization to the exotic Hill case of the chiral decomposition of Ref. [9, 10].

II. EXOTIC HILL SOLUTIONS

The planar Hill equations (1) derive from the symplectic form and Hamiltonian [4]

$$\begin{aligned} \Omega_0 &= dp^x \wedge dx + dp^y \wedge dy + 2m\omega dx \wedge dy, \\ H &= \frac{p^2}{2m} + \frac{1}{2}kx^2, \quad k = -3m\omega^2. \end{aligned} \quad (2)$$

For an exotic particle, the symplectic form is modified as

$$\Omega = \Omega_0 + \frac{\theta}{2} \varepsilon^{ij} dp^i \wedge dp^j, \quad (3)$$

while the Hamiltonian is unchanged [7]. The constant θ is the *non-commutative parameter*, as justified by the associated Poisson brackets

$$\{x^i, x^j\} = \frac{\theta}{\Delta} \varepsilon^{ij}, \quad \{x^i, p^j\} = \frac{\delta^{ij}}{\Delta}, \quad \{p^i, p^j\} = \frac{2m\omega}{\Delta} \varepsilon^{ij}, \quad (4)$$

where

$$\Delta = 1 - 2m\omega\theta \quad (5)$$

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is the [square-root of the] determinant of the “exotic” symplectic form (3).

Hamilton’s equations could be worked out and the solution found. We find it, however, more convenient to deduce them in a smarter way, presented in the next Section.

III. CHIRAL DECOMPOSITION

Following [4, 9, 10], we introduce chiral coordinates $x^i = X_+^i + X_-^i$,

$$p^1 = \alpha_+ X_+^2 + \alpha_- X_-^2, \quad p^2 = -\beta_+ X_+^1 - \beta_- X_-^1, \quad (6)$$

where the coefficients α_\pm and β_\pm are determined from the requirement that both the symplectic form and the Hamiltonian should split. Then the calculation analogous to the one in [4] yields

$$\begin{aligned} \Omega &= -\frac{\Delta}{\Gamma} \frac{m\omega}{2} dX_+^1 \wedge dX_+^2 + \frac{m\omega}{2} dX_-^1 \wedge dX_-^2, \\ H &= \frac{m\omega^2}{2} \left(X_+^1 X_+^1 + \frac{1}{4\Gamma^2} X_+^2 X_+^2 \right) - \frac{3m\omega^2}{8} X_-^1 X_-^1, \end{aligned} \quad (7)$$

where

$$\Gamma = 1 - \frac{3}{2}\theta m\omega. \quad (8)$$

For $\theta = 0$ we have $\Delta = \Gamma = 1$, and the commutative Hill case [4] is recovered.

For $k = 0$ the oscillator term is switched off, and the system reduces to the purely magnetic non-commutative Landau problem discussed in [9, 10].

IV. MOTIONS

The decomposition (7) implies the Poisson brackets

$$\{X_+^1, X_+^2\} = \frac{\Gamma}{\Delta} \frac{2}{m\omega}, \quad \{X_-^1, X_-^2\} = -\frac{2}{m\omega} \quad (9)$$

completed with $\{X_+^i, X_-^j\} = 0$, providing us with separated equations of motion,

$$\begin{aligned} \dot{X}_-^1 &= 0, & \dot{X}_-^2 &= -\frac{3}{2}\omega X_-^1, \\ \Gamma \Delta \dot{X}_+^1 &= \frac{\omega}{2} X_+^2, & \Delta \dot{X}_+^2 &= -2\Gamma\omega X_+^1. \end{aligned} \quad (10)$$

Off the critical case, $\Delta \neq 0$, the solution is therefore

$$X_-^1(t) = x_0, \quad X_-^2(t) = -\frac{3}{2}\omega x_0 t + y_0, \quad (11)$$

$$X_+^1(t) = \frac{A}{\omega^*} \sin \omega^* t - \frac{B}{\omega^*} \cos \omega^* t, \quad \omega^* = \frac{\omega}{\Delta}, \quad (12)$$

$$X_+^2(t) = \Gamma \left(2 \frac{A}{\omega^*} \cos \omega^* t + 2 \frac{B}{\omega^*} \sin \omega^* t \right), \quad (13)$$

$X_-(t)$ performs, hence, for all $\Delta \neq 0$ and Γ , a simple uniform translational motion, and can be identified with the *guiding center coordinate*. $X_+(t)$, which moves along a flattened ellipses, describes instead motions about the guiding center. Note that the X_+ dynamics depends on the non-commutative parameter θ through ω^* , while the guiding center dynamics is θ -independent. In terms of the original coordinates, the motion is

$$\begin{aligned} x(t) &= \frac{A}{\omega^*} \sin \omega^* t - \frac{B}{\omega^*} \cos \omega^* t + x_0, \\ y(t) &= \Gamma \left(2 \frac{A}{\omega^*} \cos \omega^* t + 2 \frac{B}{\omega^*} \sin \omega^* t \right) - \frac{3}{2}\omega x_0 t + y_0. \end{aligned} \quad (14)$$

see Fig. 1.

The particular case $\Gamma = 0$ is rather harmless : it simply switches off the oscillations of the second coordinate of X_+ , whereas its first coordinate still oscillates, namely with frequency $\omega^* = -3\omega$. Intuitively, we have sorts of “half-Hall motions”, see Fig. 2.

In the critical case $\Delta = 0$ i.e. for angular velocity and radius

$$\omega = \omega_c = \frac{1}{2m\theta}, \quad \text{i.e.} \quad R_c^3 = 4GMm^2\theta^2, \quad (15)$$

respectively, the system is singular, and (10) only allows $X_+(t) = 0$. Thus the elliptic motions about the guiding center are eliminated, leaving us with the guiding center motion alone, $\mathbf{x}(t) = \mathbf{X}_-(t)$ in (11). Remarkably, the latter *satisfies the Hall law*: the center-of-mass moves in the planar “magnetic” field $eB = 2m\omega$ perpendicularly to the “electric” field $e\mathbf{E} = (3m\omega^2 x_0, 0)$ with appropriate “Hall” velocity $E/B = 3\omega x_0/2$.

As seen on Fig. 1, when the critical value is approached, the rotation speeds more and more up, and, *with the exception of those whose initial conditions are consistent with the Hall law*, all motions become “instantaneous” [10].

V. SYMMETRIES

The same equations (1) also describe the symmetries. Having decomposed our system into chiral components, the symmetry plainly follows from those of our chiral solutions. For the separated equations (10) the initial conditions $\mathbf{P} = \mathbf{X}_-(0)$ and $\mathbf{K} = \mathbf{X}_+(0)$,

$$\mathbf{P} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} X_-^1(t) \\ X_-^2(t) + \frac{3}{2}\omega t X_-^1(t) \end{pmatrix}, \quad (16)$$

$$\begin{aligned} \mathbf{K} &= \begin{pmatrix} -B/\omega^* \\ 2\Gamma A/\omega^* \end{pmatrix} = \\ &\begin{pmatrix} X_+^1(t) \cos \omega^* t - \frac{1}{2\Gamma} X_+^2(t) \sin \omega^* t \\ 2\Gamma X_+^1(t) \sin \omega^* t + X_+^2(t) \cos \omega^* t \end{pmatrix}, \end{aligned} \quad (17)$$

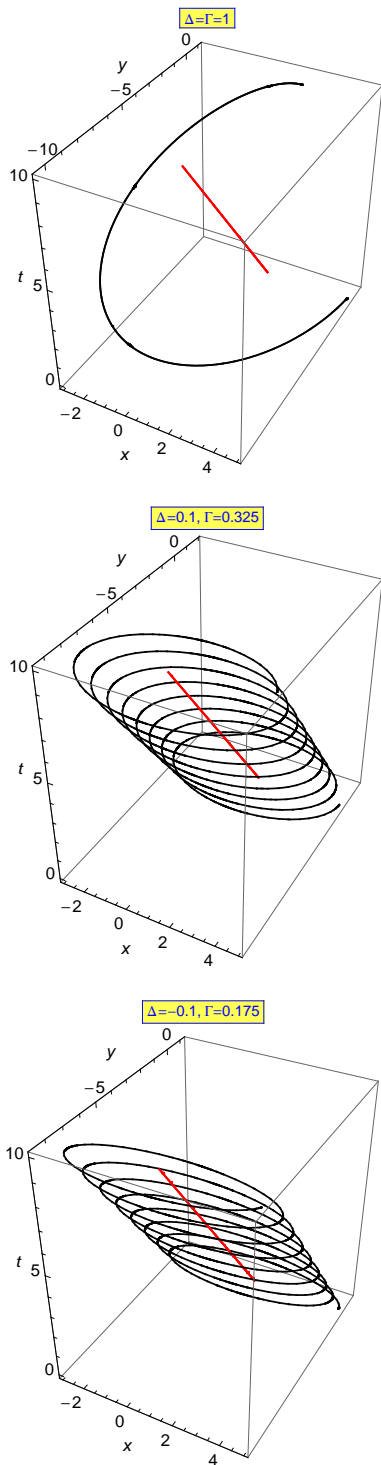


FIG. 1: Trajectories for the exotic Hill equations unfolded into time (the vertical coordinate). When the critical value $\Delta = 0$ is approached, “non-Hall” trajectories speed up, while the guiding center (in red) moves following the Hall law. The latter are the only allowed motions when $\Delta = 0$. After crossing the critical value, the sense of rotation is reversed.

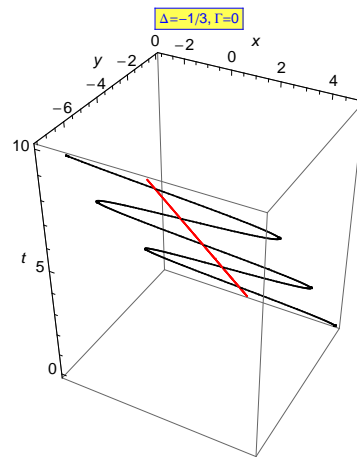


FIG. 2: For $\Gamma = 0$ the oscillations of the y coordinate disappear but $x(t)$ still oscillates around the Hall trajectory of the guiding center.

are plainly constants of the motion. They are interpreted as translations and generalized boosts, respectively. For their Lie algebra structure we find, off the critical case $\Delta \neq 0$, the commutation relations

$$\{P^1, P^2\} = -\frac{2}{m\omega}, \quad \{K^1, K^2\} = \frac{\Gamma}{\Delta} \frac{2}{m\omega}, \quad (18)$$

supplemented with $\{\mathbf{P}, \mathbf{K}\} = 0$. We have therefore *two uncoupled Heisenberg algebras with different central charges* $2/\omega$ and $2\Delta/\omega$, respectively. Adding the obvious time-translation-symmetry would provide us with exotic Newton-Hooke symmetry without rotations cf. [4].

In the critical case $\Delta = 0$ the \mathbf{X}_+ dynamics becomes trivial. $\mathbf{K} = 0$ drops out, and the symmetry algebra reduces to the single Heisenberg algebra of $\mathbf{P} = \mathbf{X}_-$ alone [plus time translations], with extension parameter determined by the non-commutative parameter, $2/\omega_c = 4m\theta$.

VI. CONCLUSION

Guided by the analogy with the non-commutative Landau problem [8, 10], we extended our previous study of Hill’s equation to exotic particles. Our most interesting result says that for a critical angular velocity i.e. for a critical radius determined by the non-commutative parameter θ , cf. (15), the only motions are those determined by the Hall law. The role of θ is to enhance the “Hall-type” behavior, eliminating all the others in the critical case $\Delta = 0$. We note that Hall motions in stellar dynamics have been considered before [11, 12].

Except for the lack of rotational symmetry due to the anisotropic oscillator term in (1), our results are reminiscent to those for the non-commutative Landau problem [7–10], and generalize those derived in [4] for $\theta = 0$.

It is worth mentioning that the dimensional drop in the critical case exemplifies the degeneration studied in [13] in a general setting.

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